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GENERIC SETS AND MINIMAL α-DEGREES

BY

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ABSTRACT. A non-α-recursive subset G of an admissible ordinal α is of minimal α-degree if every set of strictly lower α-degree than that of G is α-recursive. We give a characterization of regular sets of minimal α-degree below $0'$ via the notion of genericity. We then apply this to outline some 'minimum requirements' to be satisfied by any construction of a set of minimal $\aleph_\omega^L$-degree below $0'$.

In 1956 Spector [8] showed the existence of a minimal Turing degree. This result stimulated the study of initial segments of degrees of unsolvability (cf. Yates [9]), and the technique used in Spector's proof led Sacks to the formulation of the method of forcing with perfect closed sets (cf. Sacks [4]), a method which proved to be very important in higher recursion theory. Despite such success, the basic problem of proving the existence of a minimal α-degree, for all admissible ordinals α, remains only partially answered. The best result to date is Maass's proof [3] that minimal α-degrees exist if the Σ_2-cofinality of α ($\sigma 2\mathrm{cf}(\alpha)$) is not less than the Σ_2-projectum of α ($\sigma 2\mathrm{p}(\alpha)$). This result improves upon Shore's [6] where Σ_2-admissibility of α was assumed. Nevertheless, the solution for the minimal α-degree problem, in the general case when α is not Σ_2-admissible, remains open. Indeed it is not even known whether minimal $\aleph_\omega^L$-degrees exist ($\aleph_\omega^L$ = the ωth constructible cardinal in the sense of Gödel).

From the methodological point of view, all sets of minimal α-degree that have hitherto been produced carry two common features: the use of a short initial segment of reduction procedures and the approximation to the set desired via splitting trees and full trees. To ensure that a set G lies in the intersection of splitting trees and full trees, the reduction procedures were arranged so as to allow a decreasing sequence of such trees to converge to a set of minimal α-degree. In Shore [6] an example was given to indicate why, in the case $\alpha = \aleph_\omega^L$, his method of 'decreasing sequences of trees' breaks down. It has since become axiomatic that any new approach to tackle the minimal α-degree problem should be first tested on $\aleph_\omega^L$. We prove in this paper via a characterization theorem that roughly speaking, no radically

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different approach is required to construct a regular set of minimal α-degree below $0'$, the complete α-recursively enumerable degree, if such a set exists.

We assume that the reader is familiar with the basics of α-recursion theory (cf. Chong [1], Sacks and Simpson [5]). We will also assume known the notions of initial segments (or strings), extensions and compatibility of initial segments as defined in Shore [6].

DEFINITION. A tree is an α-recursive set of initial segments.

Note that an initial segment is an α-finite function from an ordinal less than α into $\{0, 1\}$. Also, the definition of trees in this case is more general than that given in Shore [6]; this greater generality gives us more room for manipulation.

DEFINITION. A tree T is quasi-splitting with respect to the reduction procedure $\{e\}$ if there exist two α-recursive functions f, g: $T \to \alpha$ such that for all p in $T, f(p) \geqslant g(p)$ and if p, q are in T, then $p|g(p)$ incompatible with $q|g(q)$ implies that $\{e\}^p|f(p)$ incompatible with $\{e\}^q|f(q)$.

DEFINITION. A tree T is indiscernible with respect to the reduction procedure $\{e\}$ if there exists an α-recursive function f: $T \to \alpha$ such that if p and q are in T, then

$$\{e\}^p(x) = \{e\}^q(x) \quad \text{for all } x < \min\{f(p), f(q)\}.$$

DEFINITION. A subset G of α is generic if for each reduction procedure $\{e\}$ such that $\{e\}^G$ is total, there is a tree containing G as a branch (i.e. unboundedly many initial segments of G are in T) such that T is either quasi-splitting or indiscernible with respect to $\{e\}$. Furthermore, if T is quasi-splitting, then $\sup f(p)$, $\sup g(p)$ are equal to α, where p is in $G \cap T$. Similarly, if T is indiscernible, then $\sup f(p)$ is equal to α for p in $G \cap T$.

DEFINITION. A non-α-recursive set G is of minimal α-degree if every set of strictly lower α-degree than that of G is α-recursive.

We now state and prove our characterization theorem for minimal α-degrees below $0'$, for all admissible ordinals α.

CHARACTERIZATION THEOREM. *A regular, non-α-recursive set G lying below $0'$ is of minimal α-degree if and only if it is generic.*

We will prove the theorem in a sequence of lemmas.

LEMMA 1. *If G is non-α-recursive and generic, then G is of minimal α-degree.*

PROOF. Let $\{e\}$ be a reduction procedure and suppose that $\{e\}^G$ is total. Since G is generic, there is a tree T containing G as a branch such that T is either quasi-splitting or indiscernible.

Suppose that T is quasi-splitting. We claim that $G \leqslant_\alpha \{e\}^G$. Fix an ordinal

σ less than α. We will decide $G|\sigma$ using $\{e\}^G$. By assumption, there exist α-recursive functions f and g accompanying the tree T. Choose a p in T such that $g(p) > \sigma$ and such that $\{e\}^p|f(p)$ is an initial segment of $\{e\}^G$. Such p always exists by the genericity of G. We claim that $p|\sigma = G|\sigma$. If the claim is false, take q an initial segment of G such that $g(q) > \sigma$. Now p and q are incompatible and the first x which witnesses this is less than σ. This means that $p|g(p)$ is incompatible with $q|g(q)$ and so implies that $\{e\}^p|f(p)$ is incompatible with $\{e\}^q|f(q)$. But this is not possible since both of these computations yield initial segments of $\{e\}^G$. This contradiction shows that $p|\sigma = G|\sigma$.

Suppose that T is indiscernible. We claim that $\{e\}^G$ is α-recursive. Fix an ordinal σ less than α. Locate a p in T such that $f(p) > \sigma$, where f is an α-recursive function associated with the indiscernible tree T. It is clear that $\{e\}^p|\sigma = \{e\}^G|\sigma$. From these two facts we conclude that G is of minimal α-degree.

Let O be a regular complete α-recursively enumerable set.

Lemma 2. *Let G be a regular set α-recursive in O. Then G is a tame Σ_2-set.*

Proof. Let $\{e\}^O = G$. Fix an ordinal $\sigma < \alpha$. By the regularity of G, for any α-finite set K,

$$G|\sigma = K \leftrightarrow (\exists \tau, \eta, \upsilon)(\forall \zeta)\big(\langle K, \eta, \upsilon\rangle \in R_e^\tau \text{ and } K_\eta \subseteq O^\tau \text{ and } K_\upsilon \cap O^\zeta = \varnothing\big).$$

Here $\{R_e\}_{e<\alpha}$ and $\{K_\upsilon\}_{\upsilon<\alpha}$ are respectively α-recursive listings of α-r.e. sets and α-finite sets.

The next lemma states that functions of minimal α-degree below $0'$ do not dominate all α-recursive functions. This is a crucial fact required in the proofs of Lemmas 4 and 5. Our proof of Lemma 3 is a variation of Cooper's proof of Theorem 2 in [2]. The proof is a finite injury priority argument of the unbounded type. We employ Shore's blocking technique [7] to carry out the construction.

Recall that a function B: $\alpha \to \alpha$ dominates an α-recursive function f if there is an x_0 such that for all $x > x_0$, $B(x) \geqslant f(x)$. B: $\alpha \to \alpha$ is tame Σ_2 if there exists an α-recursive function B': $\alpha \times \alpha \to \alpha$ such that for all $\sigma < \alpha$, there is a τ satisfying $B'(\upsilon, x) = B(x)$ for all $x < \sigma$ and all $\upsilon \geqslant \tau$. The Σ_2-cofinality of α is the least ordinal κ for which there is a Σ_2-definable function mapping κ unboundedly into α.

Lemma 3. *Let B: $\alpha \to \alpha$ be a tame Σ_2-function. If B is of minimal α-degree, then there is an α-recursive function which is not dominated by B.*

PROOF. Let κ be the Σ_2-cofinality of α and let α^* be the Σ_1-projectum of α. Let p: $\alpha \to \alpha^*$ be an α-recursive projection. Then p^{-1} defines a tame Σ_2-function from α^* onto α. Define

$$A(x, y) \leftrightarrow p^{-1}(y) > j(x),$$

where j: $\kappa \to \alpha$ is a Σ_2-cofinality function approximated by j': $\alpha \times \kappa \to \alpha$. A is then a Σ_2-relation on $\kappa \times \alpha^*$. By Jensen's Uniformization Theorem there is a Σ_2-function uniformizing A.[1] It is easy to see, writing ψ as the Σ_2-uniformizing function, that the image of κ under ψ is unbounded in α^*. Furthermore, ψ is tame Σ_2 since its domain is κ. Note that ψ can be an α-finite function. In any case, it is possible to make ψ and its α-recursive approximation ψ': $\alpha \times \kappa \to \alpha^*$ strictly increasing, and we assume that this has been done. We assume that B dominates every α-recursive function and, based on this assumption, construct a set C which is non-α-recursive and lying strictly below B in α-degree. Our requirements are:

$$R_e \neq C, \qquad e < \alpha; \qquad \{e\}^C \neq B, \qquad e < \alpha.$$

Stage 0. Set $C'(0, x) = 1$ for all x, and set $I(0, y) = \varnothing$ for all $y < \kappa$.

Stage $\sigma > 0$. Set $C'(<\sigma) = \{x | (\exists \tau < \sigma)(\forall v)(\sigma > v \geqslant \tau \to C'(v, x) = 0)$ and the least such τ is $\leqslant B'(\sigma, x)\}$.

Our purpose is to make x in C if and only if $C'(v, x) = 0$ for all v greater than some σ_x of value less than $B(x)$. No x will be allowed to enter C after stage $B(x)$. In this manner we make C α-recursive in B.

Compute $p|\sigma$ and compute $\psi'(\sigma, \psi)$ for all $y < \kappa$ (here we assume that $\kappa < \alpha$. If $\kappa = \alpha$ then ψ is replaced by the identity function). For each y, find all $e < \sigma$ such that $\psi'(\sigma, y) \leqslant p(e) < \psi'(\sigma, y + 1)$ and, depending on which kind of requirement e represents, either

$$R_e^\sigma | \sigma = C'(<\sigma) \quad \text{or} \quad \{e\}_\sigma^{C'(<\sigma)} | \sigma = B'(\sigma, \cdot) | \sigma$$
$$\left(\text{i.e. } \{e\}_\sigma^{C'(<\sigma)}(x) = B'(\sigma, x) \text{ for } x < \sigma\right). \qquad (*)$$

Set $I(<\sigma, y) = \{x | x \leqslant \sigma$ and $((x \notin C'(<\sigma)$ and $\sigma > B'(\sigma, x))$ or $((\exists \tau < \sigma)(\exists z \leqslant y)(\forall v)(\sigma > v \geqslant \tau \to x \in I(v, z)))) \}$.

This set $I(<\sigma, y)$ is the set of elements which we wish to exclude from C at stage σ either to make sure that C is eventually α-recursive in B or to protect computations made by requirements of higher or equal priority.

By induction, suppose that $I(\sigma, z)$ has been defined for all $z < y$. Let $C'(\sigma, < y)$ be elements which are either in $C'(<\sigma)$ or were put into C in the

[1] The referee has kindly pointed out that Jensen's Uniformization Theorem is not needed to uniformize Σ_2 predicates (when α is admissible). Namely, let $P(x, y)$ be defined by $(Ez)Q(x, y, z)$, where Q is Π_1. Then P is uniformized by

$$(Ez)(y = (z)_0 \& Q(x, (z)_0, (z)_1) \& (y)_{y<z} \sim Q(x, (y)_0, (y)_1)).$$

The last clause is equivalent to a Σ_1 formula by admissibility.

course of defining $I(\sigma, z)$ for some $z < y$. Consider the set $N(\sigma, y)$ of requirements e satisfying $\psi'(\sigma, y) \leqslant p(e) < \psi'(\sigma, y+1)$ and (*) when $C'(<\sigma)$ is replaced by $C'(\sigma, <y)$ in the equations. Find two initial segments p and q satisfying the following conditions ((i)–(v) for all e in $N(\sigma, y)$ of type $R_e \neq C$):

(i) p and q both have length $\sigma + 1$ and extend $C'(\sigma, <y)$;

the following holds true for p replaced by q:

(ii) $p(x) = 0$ implies that $x \in C'(\sigma, <y)$ or $\sigma \leqslant B'(\sigma, x)$;

(iii) $p(x) = 0$ implies that x is not in $I(\sigma, z)$ for all $z < y$, nor is x in $I(<\sigma, y)$;

(iv) $p(x) = 0$ implies that x has not been excluded from C by any of the computations involving some requirement e, $\psi'(\sigma, y) \leqslant p(e) < \psi'(\sigma, y+1)$, which is not in $N(\sigma, y)$;

(v) $R_e^\sigma | \sigma + 1 \neq p$;

and finally if e is the least in $N(\sigma, y)$ such that e is the requirement $\{e\}^C \neq B$, then

(vi) $\{e\}_\sigma^p | \sigma + 1 \neq \{e\}_\sigma^q | \sigma + 1$.

If p and q exist, take the one such that the computation under (vi) is not equal to $B'(\sigma, \cdot)|\sigma + 1$. Suppose that this is p. Set $C'(\sigma, x) = 0$ if $x \in C'(\sigma, <y)$ or $p(x) = 0$. Set $I(\sigma, y) = \{x | x \in I(<\sigma, y) \text{ or } p(x) = 1\}$. If no p and q exist to satisfy (vi), find a p such that $p(x) = 0$ only if required by (i)–(v) and set $C'(\sigma, x)$ and $I(\sigma, y)$ again as above. Once this is done, go to $y + 1$. This completes our construction at stage σ. Finally, we set $C = \{x | (\exists \tau)(\forall \sigma \geqslant \tau)(C'(\sigma, x)) = 0\}$.

Claim 1. $C \leqslant_\alpha B$. Fix an ordinal σ. Since B is tame Σ_2, we see that $\sup B(x)$, $x < \sigma$, is bounded below α. Our construction stipulates that no $x < \sigma$ can enter C permanently after stage $\sup B(x)$, $x < \sigma$. Furthermore if $x < \sigma$ enters C at a stage before $\sup B(x)$, $x < \sigma$, and is not removed before the same stage, then it stays in C permanently. This proves Claim 1.

Claim 2. For each e, $R_e \neq C$ and $\{e\}^C \neq G''\alpha$.

The proof is by induction on blocks $\text{blk}(y) = \{e | \psi(y) \leqslant p(e) < \psi(y+1)\}$. Fix y. Assume that there is a stage σ_y such that whenever $p(e) < \psi(y)$, then

$$\begin{aligned} &R_e^\sigma | v \neq C'(<\sigma)|v && \text{for some } v < \sigma_y \text{ and} \\ & && \text{for all } \sigma > \sigma_y\text{; and} \\ &\{e\}_\sigma^{C'(<\sigma)} | v \neq B'(\sigma, \cdot)|v && \text{for some } v < \sigma_y \text{ and} \\ & && \text{for all } \sigma > \sigma_y. \end{aligned} \tag{**}$$

Let us assume further that ψ' has attained its final value for $y + 1$ by stage σ_y and that $I(\sigma, z) = I(\sigma_y, z)$ for all $z < y$ and $\sigma > \sigma_y$. We claim that there is a stage σ_{y+1} such that for all e in $\text{blk}(y)$, (**) holds with σ_y replaced by σ_{y+1}.

Suppose that no stage σ^* exists such that when σ_y is replaced by σ^* in (**), all e in the set $K = \{e | e \in \text{blk}(y)$ and e is the requirement $R_e \neq C\}$ satisfy (**). Now K is an α-r.e. set such that $\{p(e) | e \in K\}$ is an α-r.e. set bounded below α^*. Since the latter set is α-finite, so is K. Define $f(v)$ = the least σ such that $R_e^\sigma | v = C'(<\sigma) | v$, for some e in K.

By our assumption, f is unbounded in α and is α-recursive. Since B dominates every α-recursive function, there is an v_0 such that for $v > v_0$, $B(v) > f(v)$.

Now between stages $f(v)$ and $B(v)$, it is possible according to our construction to make $R_e^{\tau(v)} | v \neq C'(<\tau(v)) | v$, where $f(v) < \tau(v) < B(v)$. Our construction will attempt to preserve this computation unless at some later stage $\tau'(v) > \tau(v)$ one discovers that $R_e^{\tau'(v)}$ changes value and again restores equality. Certainly if $\tau'(v) < B(v)$ for all but α-finitely many v's, one could again readjust C to obtain inequality. If this readjustment is done, it will be permanent (thereby obtaining a σ^*) because elementhood about R_e can change mind only once (i.e. from $x \notin R_e$ to $x \in R_e$). The only hindrance to making such an adjustment, however, would be when $\tau'(v) > B(v)$ for unboundedly many v's. This cannot happen since otherwise $v \mapsto \tau'(v)$ is an α-recursive function not dominated by B.

Next suppose that no τ^* exists such that $K = \{e | e \in \text{blk}(y)$ and e is the requirement $\{e\}^C \neq B\}$ satisfies (**) when σ_y there is replaced by τ^*. Again it is immediate that K is an α-finite set. We claim that for each v, there is a $\tau(v) > v$ and an e in K such that for some $\sigma(v) \geqslant \tau(v)$

$$\{e\}_{\sigma(v)}^{C'(<\sigma(v))} | \tau(v) \neq B'(\sigma(v), \cdot) | \tau(v) \text{ or} \qquad (***)$$

the left-hand side is not total.

The claim is obviously true since its negation implies that for all v greater than some fixed v_0, computations using $C'(<\sigma)$, for any $\sigma > v_0$, on the initial segment v through any reduction procedure e in K always yield the same results as computing B'. This immediately shows that B is α-recursive (since B is tame Σ_2 and therefore $B'' v_0$ is α-finite), which is impossible since B is of minimal α-degree.

Let σ^* be as defined above and without loss of generality assume that $\sigma^* > \sigma_y$. Define

$$f(v) = \inf \text{ (over } e \text{ in } K') \text{ of } \sup C'(<\sigma(v)),$$

where K' is that subset of K consisting of all e satisfying the first part of the disjunction (***). Clearly K' is an α-r.e. subset of K and therefore α-finite. Thus f is an α-recursive function. By assumption on B, there is an v_0 such that for all $v > v_0$, $B(v) > f(v)$. Set

$$K'' = \{e | (\exists \tau(v) > v_0)(\exists \sigma(v) > \tau(v))(\exists p, q)$$
$$\text{(length of } p, q = \sigma(v) \text{ and } p, q$$
$$\text{satisfy (vi) of our construction at stage } \sigma \geqslant \sigma(v))\}.$$

This is an α-finite set. Notice that it is possible to retain an inequality in the computation of $\{e\}^C \neq B$, for each e in K'', at any stage $\sigma > v_0$ by switching between p and q. Now to indicate the dependence of $\tau(v)$ and $\sigma(v)$ on e, let us write instead $\tau(e, v)$ and $\sigma(e, v)$. Then clearly $\sup \tau(e, v)$, for e in K'', is bounded below α. Using the fact that B is tame Σ_2, our construction therefore guarantees that there is a stage $v_1 > v_0$ such that for all $\sigma \geqslant v_1$, for all e in K'',

$$\{e\}_\sigma^{C'(<\sigma)} | v_1 \neq B'(\sigma, .) | v_1.$$

On the other hand, any e in $K - K''$ will satisfy one of the following three conditions:

(a) for some $v > v_0$, $\{e\}_{\sigma(e, v)}^{C'(<\sigma(e, v))} | \tau(e, v) \neq B'(\sigma(e, v), .) | \tau(e, v)$;

(b) for some $v \leqslant v_0$, a computation as (a) occurs;

(c) $\{e\}_\sigma^{C'(<\sigma)} | v = B'(\sigma, .) | v$ for all σ and the computation on the left-hand side is not defined for input $x = v$ (here we allow $v = \alpha$).

It is easy to see that the e's in (a) and (b) each form an α-finite set, and therefore so do the e's in (c). Now $s = \sup \tau(e, v)$, for e in (a), is bounded below α. There is therefore a stage v_a such that for all $\sigma \geqslant v_a$, $B'(\sigma, .) | s = B | s$. By the fact that e is in (a), no elements put into C by stage $\sigma(e, v)$ will be discarded; hence computations that show inequality will stay. This inequality will change into equality only if B' changes its values on the initial segment s. In any case, by stage v_a, computations on both sides settle down and so we are left for each e in (a) with either a permanent inequality of equality. In the latter situation, we may regard e as being in (c), keeping in mind of course the stage v_a.

Let now $s = \sup \tau(e, v)$, for e in (b) not in (a). Again s is bounded below α. Let v_b be a stage where for all $\sigma \geqslant v_b$, $B'(\sigma, .) | v_0 = B | v_0$ and $B'(\sigma, .) | s = B | s$. By this stage, all x in $C'(< \sigma(e, v))$ which ought not be present in C would have been discarded permanently. Then $\{e\}^{C'(<v_b)} | \tau(e, v)$ is either not total, or equal to $B | \tau(e, v)$, or not equal to $B | \tau(e, v)$. In the first two instances, we may consider the e to be in (c), keeping in mind the stage v_b.

Finally, if there is no stage v_c such that if e is in (c), then the corresponding v that expresses equality in the computation is less than v_c; and then it is straightforward to show that B is α-recursive, which is of course not true.

Now let $\tau^* = \sup\{v_1, v_a, v_b, v_c\}$. Let $\sigma_{y+1} = \sup\{\sigma^*, \tau^*\}$, and we see that this is the uniform stage that we require.

Now the function $y \mapsto \sigma_y$ is a Σ_2-function defined on κ. Thus for $y < \kappa$, sup σ_z, $z < y$, is always bounded below α. We conclude that induction can be carried out through κ and this proves Claim 2, and therefore Lemma 3.

LEMMA 4. *Let G be regular and of minimal α-degree below $0'$ and let $\{e\}^G$ be total and α-recursive. Then there is an indiscernible tree T containing G as a branch.*

PROOF. Let $\{e\}^G = R$ be α-recursive. Define

$$B^* = \left\{(\gamma_1, \gamma_2, \gamma_3) \middle| \{e\}^{G|\gamma_2}|\gamma_1 \text{ is defined in } \gamma_3 \text{ steps and } \gamma_2 \geqslant \gamma_1\right\}.$$

Then $B^* <_\alpha G$. By the minimality of G, B^* is either α-recursive or of the same α-degree as G.

Now for each γ, let $B(\gamma)$ be the least pair (γ_2, γ_3) such that $(\gamma, \gamma_2, \gamma_3)$ is in B^*. Then $B \leqslant_\alpha B^*$. Suppose that B is of the same α-degree as G, so that Lemma 3 can be applied on B. Note that B is regular.

By Lemma 3, there is an α-recursive function which is not dominated by B. Thus there is an α-recursive ψ such that $\psi(\gamma) > B(\gamma)$ for unboundedly many γ's. If, on the other hand, B is α-recursive, then there is also an α-recursive function ϕ which satisfies the same condition. Our purpose is to make use of ψ or ϕ, as the case may be, to 'uniformize' computations of the reduction procedure $\{e\}$ using G as an oracle. We now construct an indiscernible tree T containing G as a branch. The construction relies heavily on the 'uniformizing effect' of the function ψ (or ϕ). From now on, we assume that we have the function ψ at our disposal. Similar argument can be used if we have the function ϕ.

Stage 0. Set $T_0 = \varnothing$, and $f(\varnothing) = 0$.

Stage $\sigma > 0$. We let T_σ consist of all p's in T_τ, $\tau < \sigma$, and also all initial segments p which satisfy:

(i) length $(p) = \sigma$.

(ii) There is a $\gamma_1 \leqslant \sigma$ and a γ_3 such that $\psi(\gamma_1) > (\sigma, \gamma_3)$.

(iii) $\{e\}^p|\gamma_1 = R|\gamma_1$, and the computation completes in γ_3 steps.

Define $f(p) = \gamma_1$.

Let T be the union of all T_σ. Then f is an α-recursive function with domain T. It is clear that T is an α-recursive tree.

We claim that G is a branch of T. If not, there is a z such that whenever p in T is an initial segment of G, then the length of p is less than z. Let γ_1 be so large that the least γ_2-satisfying $(\gamma_1, \gamma_2, \gamma_3)$ in B, for some γ_3, is of length greater than z. There is then a $\sigma \geqslant \gamma_2$ such that for some $\rho_1 \geqslant \sigma$, $\psi(\rho_1)$ is larger than the least pair (ρ_2, ρ_3) such that (ρ_1, ρ_2, ρ_3) is in B. At stage ρ_2, $G|\rho_2$ would then be included in T_{ρ_2}. Since $\rho_2 \geqslant \rho_1 \geqslant \sigma \geqslant \gamma_2 \geqslant z$, we have a contradiction.

It is now immediate that T is an indiscernible tree. Since G is a branch of T, we see that sup(p), p in $G \cap T$, is α. Let p and q be in T. Then they are in

T because, among other things, they satisfy (iii) of the construction. Since $R = \{e\}^G$, we see that $\{e\}^p$ and $\{e\}^q$ agree on the initial segment γ, where $\gamma = \min\{f(p), f(q)\}$.

The following lemma completes our proof of the characterization theorem.

LEMMA 5. *Let G be regular and of minimal α-degree below $0'$. If $\{e\}^G$ is total and G is α-recursive in $\{e\}^G$, then there is a quasi-splitting tree T containing G as a branch.*

PROOF. Let $F = \{e\}^G$ and let $G = \{d\}^F$.

For convenience, let $e[G, \gamma_1, \gamma_2]$ denote computations of $\{e\}$ on the initial segment γ_2 using information from $G|\gamma_1$. Define

$$B = \{(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4) | G|\gamma_0 \subset \{d\}^{e[G,\gamma_1,\gamma_2]}|\gamma_3 \subset G|\gamma_1 \text{ in less than } \gamma_4 \text{ steps, and } \gamma_2, \gamma_3 < \gamma_1\}.$$

Then $B \leqslant_\alpha G$ and by the minimality of G, B is either α-recursive or of the same α-degree as G. Note that for each γ_0, there is a quadruple $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ such that $(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is an element of B. Thus by Lemma 3, there is an α-recursive function ψ such that for unboundedly many γ_0's, $\psi(\gamma_0)$ is bigger than the least quadruple $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ satisfying $(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ in B.

Let p be an initial segment of length γ_1 and denote by $e[p, \gamma_1, \gamma_2]$ the computations of $\{e\}$ on the initial segment γ_2 using p as an oracle. Define

$$B' = \{(p, \gamma_1, \gamma_2, \gamma_3, \gamma_4) | \text{ length}(p) = \gamma_1 \text{ and } \{d\}^{e[p,\gamma_1,\gamma_2]}|\gamma_3 \subset p \text{ in } \gamma_4 \text{ steps, and } \gamma_2, \gamma_3 < \gamma_1\}.$$

Note that B' is α-recursive. We will now construct the required tree T containing G as a branch. The construction will go by stages.

Stage 0. $T_0 = \varnothing$.

Stage $\sigma > 0$. T_σ consists of all p's in T_τ, $\tau < \sigma$, and p's satisfying I or II below:

(I)(i) length $(p) = \sigma$.

(ii) There is a $\gamma_0 \leqslant \sigma$ and $\gamma_2, \gamma_3 \leqslant \sigma$ such that $\psi(\gamma_0) > (\sigma, \gamma_2, \gamma_3, \gamma_4)$ for some γ_4. Furthermore, $(p, \sigma, \gamma_2, \gamma_3, \gamma_4)$ is in B'.

(iii) There is a sequence of compatible initial segments $\{q_\upsilon\}_{\upsilon<\zeta}$, $\zeta \leqslant \sigma$, such that denoting q as the union of this sequence, one has $p \cap q \neq \varnothing$.

(iv) $\{d\}^{e[p,\sigma,\gamma_2]}|\gamma_3$ is incompatible with q.

(v) $\{d\}^{e[p,\sigma,\gamma_2]}|\gamma_3$ is incompatible with all $\{d\}^{e[p',\sigma,\rho_2]}|\rho_3$ (denoted (*)), where p' is in T_τ, $\tau < \sigma$ (hence $\rho_2, \rho_3 \leqslant \tau < \sigma$) and (*) is incompatible with q.

(II)(i) length $(p) = \sigma$.

(ii) There exist $\gamma_0 \leqslant \sigma$ and γ_2, γ_3 such that $\psi(\gamma_0) > (\sigma, \gamma_2, \gamma_3, \gamma_4)$ for some γ_4. Furthermore, $(p, \sigma, \gamma_2, \gamma_3, \gamma_4)$ is in B'.

(iii) $\{d\}^{e[p,\sigma,\gamma_2]}|\gamma_3 \supset q$, where q is as in (I)(iii), except that q is allowed to be $\varnothing$.

(iv) There is a p' also satisfying (i), (ii) and (iii) of (II). Thus length $(p') = \sigma$, and there exist $\tau_0, \tau_2, \tau_3, \tau_4$ replacing $\gamma_0, \gamma_2, \gamma_3, \gamma_4$ for p above. Furthermore,

$$\{d\}^{e[p,\sigma,\gamma_2]}|\gamma_3 \text{ is incompatible with } \{d\}^{e[p',\sigma,\tau_2]}|\tau_3.$$

Let T be the union of T_σ, $\sigma < \alpha$. We observe that an initial segment of length σ is in T if and only if it is in T_σ. Since our construction of T_σ is effective, it implies that T is α-recursive.

We now show that G is a branch of T. Suppose for the sake of contradiction that if p in T is an initial segment of G, then the length of p is less than an ordinal z, which is fixed throughout. Let p^* be an initial segment of G of length ζ bigger than z. There is an $x > \zeta$ and a $\gamma_0 > x$ such that the least quadruple $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ satisfying $(\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ in B' is less than $\psi(\gamma_0)$. This means that by stage γ_1 at the latest one is able to find $p_1, \gamma_2, \gamma_3, \gamma_4$ such that $(p_1, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is in B. Furthermore p_1 is an initial segment of G of length γ_1. Indeed the following is true:

$$p^* \subset G|\gamma_0 \subset \{d\}^{e[p_1,\gamma_1,\gamma_2]}|\gamma_3 \subset G|\gamma_1 = p_1.$$

There are now two possibilities:

(1) In T there is a q which properly extends all p's in T which are initial segments of G. Furthermore q is not an initial segment of G. Then length (q) is bigger than z. Since p_1 is not in T, hence not in T_{γ_1}, it implies that at stage γ_1,

> $\{d\}^{e[p_1,\gamma_1,\gamma_2]}|\gamma_3$, while incompatible with q,
> is compatible with a certain computation
> $\{d\}^{e[p',\tau_1,\tau_2]}|\tau_3$ (denote this by $(+)$). And
> $(+)$ is incompatible with q.

This is true by virtue of (I). But since $(*)$ is incompatible with q which we know extends all initial segments of G which are in T, we see that $(*)$ also properly extends all initial segments of G in T, and, since it is compatible with $\{d\}^{e[p_1,\gamma_1,\gamma_2]}|\gamma_3$, is an initial segment of G of length larger than z. It then follows that G and p' agree on an initial segment larger than z. Since p' is in T, we have a contradiction.

(2) There is no initial segment in T which properly extends p's in G and T. Let q be the union of all p's which are in G and T. Then the length of q does not exceed z. The fact that p_1 and all other initial segments of G longer than z do not enter T implies (by (II)) that

if $\{d\}^{e[r,\rho_1,\rho_2]}|\rho_3 \supset q$ and
$\{d\}^{e[r',\rho_1',\rho_2']}|\rho_3' \supset q$, then the
two computations agree on $x < \{\min \rho_3, \rho_3'\}$,
whenever r and r' properly extend q
and are incompatible.

One concludes from this that if $x > z$, then x is in G if and only if there is an r, a ρ_1 = length (r), two ordinals ρ_2 and ρ_3 not exceeding ρ_1, such that $\rho_3 > x$ and $\{d\}^{e[r,\rho_1,\rho_2]}|\rho_3$ is defined in some ρ_4 stage and the computations with input x yield 0 as output. Since $\{e\}^G$ is total, initial segments r and their related ordinals can always be found, and so we conclude that G is α-recursive, which is again a contradiction. Therefore G is a branch of T.

Finally, we define two α-recursive functions f and g in the following manner: let p be in T_σ and length $(p) = \sigma$. Then indeed σ is the stage where p first occurs in T, and so there exist ordinals $\gamma_2, \gamma_3 \leqslant \sigma$ associated with p in the manner that we have described in the construction of T above. Choose and fix once and for all these two ordinals (carried out in an effective manner). Define $g(p) = y_3$ and define $f(p) = \max\{\gamma_2, \gamma_3\}$. It is now clear that T together with the α-recursive functions f and g form a quasi-splitting tree.

Now Lemmas 1, 4 and 5 together prove the Characterization Theorem. We suspect that the theorem can be generalized to minimal α-degrees incomparable with 0′, i.e. one probably does not need the degree 0′ to prove Lemma 3. It is obvious that generalizing Lemma 3 within the realm of definability (i.e. for Σ_n definable sets B) will be combinatorially much more intricate, and the lemma is probably false for large n–if α is 'far' from being Σ_n-admissible. On the other hand, it is worth noting that with the absence of Lemma 3, one can prove a weaker version of Lemmas 4 and 5 where trees T are only α-r.e., provided that G is regular and hyperregular. This is true by virtue of the fact that if B is a regular, hyperregular function of minimal α-degree, then B does not dominate the partial α-recursive function ψ whose domain is a regular set of complete α-r.e. degree with complement of order-type Σ_2-cofinality of α, and whose range is α. We shall not go into the details.

This paper was motivated by the minimal α-degree problem for admissible ordinals α which are not Σ_2-admissible. We have shown that for these α's, there is a 'correct application' of the method of 'convergence via trees' which would produce regular sets of minimal α-degree–if indeed such sets exist. Thus if G is such a set, then there is a (partial) one-to-one function P: $\alpha \to \alpha$ such that whenever $\{e\}^G$ is total, then $P(e)$ is an index of an α-recursive tree, containing G as a branch, which is either indiscernible or quasi-splitting with respect to the reduction procedure $\{e\}$. Conversely, if such a P exists, then any set in the intersection of trees with indices in range (P) is a set of minimal α-degree (assuming that there is nonempty intersection).

For $\alpha = \aleph_\omega^L$, and for many other α's, the minimal α-degree problem is then equivalent to one of searching for the 'proper' P which admits a judicious ordering of priority of reduction procedures (towards the goal of achieving nonempty intersection, among other things). In this respect, one could roughly infer the 'minimum requirements' that any such P should satisfy:

(1) P cannot be tame, in the sense that not every α-finite set K has an α-finite image under P. To see this, consider the reduction procedures (cf. Shore [6])

$$\{n\}^X(x) = \begin{cases} X(x) & \text{if } L_x \models \text{`there are less} \\ & \text{than } n \text{ cardinals'}, \\ 1 & \text{otherwise.} \end{cases}$$

For each n, $P(n)$ is an indiscernible tree T_n and G is in the intersection of the T_n's. If $\{P(n)\}_{n<\omega}$ is α-finite, then G is easily checked to be α-recursive, which is a contradiction.

(2) The complete α-recursively enumerable degree is α-recursive in the α-degree of P. This is immediate using (1) and the fact that the set

$$K = \{(n, m) | \upsilon m \text{ is the least integer such that } P(n) < \aleph_m^L\}$$

is α-finite. Take O as defined earlier, and we see that O is α-recursive in P using K as parameter.

(3) It is however true that the set G constructed via P must be tame, in the sense that if R_e denote the eth-α-recursively enumerable set, then

$$K_m = \{(e, \sigma) | \aleph_m^L > e \text{ and } \sigma \text{ is the least ordinal } \tau \text{ such that } G(\tau) \neq R_e(\tau)\}$$

is an α-finite set for all m. Otherwise, there exist an m, an α-finite sequence $\{e_n\}_{n<\omega}$ and a sequence $\{\sigma_n\}_{n<\omega}$, such that

$$(e_n, \sigma_n) \text{ in } K_m \text{ and } \sigma_n > \aleph_{n+1}^L \text{ for all } n.$$

Then we have

$$x \text{ in } G \leftrightarrow \text{there exist } \sigma \text{ and } n \text{ such that } x \text{ is in } R_{e_n}^\sigma \text{ and}$$
$$L_\sigma \models \text{`there are not more than } n \text{ cardinals'}.$$

Since this implies that G is α-recursively enumerable, and therefore not of minimal α-degree (cf. Chong [1] and Shore [7]), we have a contradiction.

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